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ON BLOCKING NUMBERS OF SURFACES

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Abstract

The blocking number of a manifold is the minimal number of points needed to block out lights between any two given points in the manifold. It has been conjectured that if the blocking number of a manifold is finite, then the manifold must be flat. In this paper we prove that this is true for 2-dimensional manifolds with non-trivial fundamental groups.

1 Introduction

It has been asked whether how lights on a manifold being blocked determines the geometry of the manifold. In this paper we prove a conjecture about blocking numbers for the 2-dimensional case.

Throughout the whole paper, we let (M, g) to be a smooth, closed, orientable 2-dimensional Riemannian manifold, where g is the Riemannian metric. By a *geodesic segment* we mean a geodesic $\gamma : [0, a] \to M$ where a is the length of γ.

Given two points on (M, g), we can connect them by geodesic segments. We define the *blocking number* between the two points to be the minimal number of points needed to block all geodesic segments connecting them. The blocking number of the manifold (M, g) is then defined to be the supremum of the blocking numbers between any two points in (M, g). In some literatures, this number is referred as the *security threshold* of (M, g). A manifold with finite blocking number is said to be *secure*.

It is suggested that the blocking number of a Riemannian manifold could give some information about the geometry of that manifold. A general conjecture is that if (M, g) has finite blocking number, then the metric g must be flat. Our task in this paper is to show that this conjecture is true for orientable surfaces with non-trivial fundamental groups.

Similar results for 2-dimensional tori have been obtained independently by V. Bangert and E. Gutkin in a preprint posted on ArXiv on 22 Jun 08, using an analogous but a somewhat different method, please see [4].

2 Preliminaries and Previous Results

Let us assume that (M, g) is a smooth, complete, compact orientable Riemannian manifold. We also assume that geodesics are parametrized by arc length. If $\gamma : [s_0, s_1] \to M$ is a geodesic segment, we call $\gamma(s_0)$ and $\gamma(s_1)$ the *endpoints* of γ and all other points of γ the *interior points* of γ. We say a geodesic segment *connects* two points x and y if the points are the endpoints of γ. A geodesic γ is said to be *blocked* by a point z if z is an interior point of γ.

Given any two points in (M, g), we can define the blocking number between them as follows:

Definition 2.1 *Let x and y be two points in (M, g). The blocking number $B(x, y)$ between x and y is a positive integer (could be infinite) that is the minimal number of points needed to block away all geodesic segments connecting x and y.*

Next we can give the definition of the blocking number of a manifold.

Definition 2.2 *The blocking number of a Riemannian manifold (M, g), denoted by $B(M, g)$, is the supremum of the blocking numbers between any two points in M. i.e.,*

$$B(M, g) = \sup\{B(x, y) |\ x, y \in M\}$$

Example 1: Let (M, g) be a Hadamard manifold, which means (M, g) is simply connected and has non-positive sectional curvature everywhere, this way we have $B(M, g) = 1$. This is because given any two points in (M, g), the Cartan-Hadamard theorem implies that these two points are connected by a unique geodesic segment.

Example 2: Let (M, g) be a flat torus, then $B(M, g) = 4$. To see why this is the case, let us first remark that blocking number is invariant under affine transformation, therefore we can assume that (M, g) is the standard flat torus given by $\mathbb{R}^2/\mathbb{Z}^2$. Now we want to show that for any two points $x, y \in \mathbb{R}^2/\mathbb{Z}^2$ we have $B(x, y) = 4$. Without loss of generality, let us assume x be a point in M and $\tilde{x} \in \mathbb{R}^2$ with $\tilde{x} = (0, 0)$, such that $\tilde{x}$ projects to x. Now if y is any other point in M with coordinates $(a, b) \in [0, 1] \times [0, 1]$, then y is lifted to points in $\mathbb{R}^2$ with coordinates $(a+m, b+n)$, $m, n \in \mathbb{Z}$. Let us connect each $(a+m, b+n)$ to $\tilde{x}$ by a straight segment $\tilde{\gamma}_{m,n}$. It is easy to see that the projections of $\tilde{\gamma}_{m,n}$ to M coincide with all geodesics connecting x and y. Next, for each (m, n) let us consider the midpoint of $\tilde{\gamma}_{m,n}$ to be its the blocking point. These blocking points projects to M having coordinates $(\frac{a}{2}, \frac{b}{2}) + (\frac{1}{2}\mathbb{Z})^2/\mathbb{Z}^2$. It is easy to see

there could be at most 4 of such blocking points. Therefore we conclude that the set of geodesics connecting x and y can be blocked by 4 points, since x and y are arbitrary, the blocking number of M is 4.

Example 3: Let (M,g) be a standard $n-sphere$, then $B(M,g)=\mathfrak{c}$, where $\mathfrak{c}$ stands for the cardinality of the continuum. This is because if x and y are two non-antipodal points, then they are connected by exactly two geodesics, such that these geodesics form a great circle. Now if x and y are two antipodal points, then they are connected by a family of distinct geodesics, each of which is a half great-circle. This family of geodesics has cardinality of $\mathfrak{c}$ and so $B(M,g)=\mathfrak{c}$.

The consideration of blocking number could originate from the study of polygonal billiard systems and geometric optics. One interesting question is that how the geometry of the manifold could relate to its blocking number. To begin with the discussions, let us remark a result that states flat manifolds have finite blocking numbers [**?**]:

Theorem 2.3 *Compact flat manifolds have finite blocking numbers.*

A natural question to ask is if the converse of the above theorem is true.

Conjecture 2.4 *A compact Riemannian manifold has finite blocking number if and only if it is flat.*

In [5], K. Burns and E. Gutkin have obtained a partial solution to the conjecture. They have related the blocking properties with metric entropy of the geodesic flow.

Theorem 2.5 *If M is a manifold without conjugate points, and the geodesic flow of M has positive metric entropy, then $B(M)$ is not finite.*

Using this theorem, we can obtain the following result for manifolds with everywhere non-positive curvature [7].

Corollary 2.6 *Let (M,g) be a Riemannian manifold of non-positive curvature. If the blocking number of (M,g) is finite, then (M,g) is flat.*

Proof: Assume that (M,g) has non-positive curvature and is not flat, then (M,g) has no conjugate or focal points. According to Pesin [10], this means the geodesic flow of (M,g) has positive entropy, hence by Theorem 2.5, the blocking number of M is not finite.$\square$

However the assumption of having no conjugate points is rather strong. For instance if M is a torus and we assume that M has no conjugate point, this

automatically means that M is flat by the Hopf's conjecture [2] and the blocking number cannot play a role here.

3 The Main Result

We are now ready to proceed to the main theme of this paper. Our main result is that Conjecture 2.4 is true for two dimensional surfaces with non-trivial fundamental groups:

Theorem 3.1 *Let (M, g) be a compact, complete, orientable 2-dimensional Riemannian manifold with $\pi_1(M) \neq \{0\}$, then (M, g) has finite blocking number if and only if (M, g) is a flat torus.*

In view of Theorem 2.3, we only need to prove the only if part of the theorem. First of all, let us recall the following classification theorem for compact orientable 2-dimensional Riemannian manifolds.

Theorem 3.2 *The homeomorphism classes of 2-dimensional manifolds are determined by the genus.*

Therefore what we will do is to separate the surfaces according to their genus. In the following we will prove some key lemmas and then we will use the lemmas to investigate surfaces of different genus.

4 Key Tools and Lemmas

We now provide all notations and lemmas needed to prove Theorem 3.1. Let us start by recalling some fundamental concepts in Riemannian geometry.

Recall that (M, g) is a geodesic complete, locally compact Riemannian manifold. If we denote by $L_g(\gamma)$ the length of the curve γ, We can define the following metric $d_g(\cdot, \cdot)$ on M:

$$d_g(x, y) = \inf_{\gamma}\{\gamma : [a, b] \to M \text{ continuous}, \gamma(a) = x, \gamma(b) = y\}$$

In this way (M, d_g) can be considered a complete metric space. A continuous curve in M is said to be *distance minimizing* if the distance in terms of d_g between the end points is equal to the length of the curve. Next, we define a *minimal geodesic* to be a continuous curve that is distance minimizing on its subsegments:

Definition 4.1 *A minimal geodesic is a geodesic segment $\gamma : [a, b] \to M$ such that for all $s, s' \in [a, b]$ we have,*

$$d_g(\gamma(s), \gamma(s')) = |s - s'|$$

In particular, the Arzela-Ascoli theorem, completeness and compactness of (M, d_g) gives us the following version of the Hopf-Rinow theorem:

Theorem 4.2 *Any two points of M can be connected by a minimal geodesic.*

It is easy to see that d_g is realized by the length of the shortest minimal geodesic, therefore we can call d_g the *geodesic distance* on M.

Now if we consider closed curves, i.e. $\gamma : [a, b] \to M$ satisfying $\gamma(a) = \gamma(b)$. A closed curve that is a geodesic is called closed geodesic or periodic geodesic. A periodic geodesic is minimal if it unwraps to a minimal geodesic.

Definition 4.3 *Let γ be a periodic minimal geodesic and x be a point not on γ, then the* distance *between x and γ, denoted by $d(x, \gamma)$, is the shortest geodesic distance between them as subsets, i.e.*

$$d(x, \gamma) = \inf\{d_g(x, y) | y \in \gamma\}$$

Definition 4.4 *Let γ be a periodic minimal geodesic and c be a minimal geodesic. c is said to be* asymptotic *to γ if c does not intersect γ and for all $\epsilon > 0$, there exists $s_0 > 0$ such that*

$$d(c(s), \gamma) < \epsilon, \text{ whenever } s > s_0$$

The first lemma is a simple fact in differential geometry:

Lemma 4.5 *Two minimal geodesics originating from the same point will not be minimal beyond their first point of intersection.*

For a proof of the lemma, see for instance, [6].

We want to relate the blocking number of M with periodic minimal geodesics. Next we will present a crucial lemma that shows the condition such that M must have infinite blocking number. Let us first remark that if γ is a non-trivial periodic minimal geodesic, then for any sufficiently small $\epsilon > 0$, the set

$$U_\epsilon(\gamma) := \{x \in M | d(x, \gamma) \leq \epsilon\}$$

has γ as its retract. Note that $U_\epsilon(\gamma)$ is diffeomorphic to an annulus. So if we consider the universal cover $\tilde{M}$ of M, $U_\epsilon(\gamma)$ is lifted to a infinite strip. Also γ will be lifted to an infinitely long minimal geodesic $\tilde{\gamma}$.

Lemma 4.6 *If M has a closed minimal geodesic and another minimal geodesic asymptotic to it, then M has infinite blocking number.*

Proof: Suppose the contrary is true, that if x, y are two points in M, we have $B(x, y) < \infty$. We show that this could lead to a contradiction by demonstrating that there is a point x on the asymptotic minimal geodesic c stated above and a point y on the closed geodesic γ, such that the blocking number between x and y cannot be finite. To prove this claim, let y be any point on γ, then consider the lifts $\tilde{\gamma}$ of γ. y would be lifted to a countable set of points y_i in a long strip. For any given point x on c we can connect x to y_i by a minimal geodesic $\tilde{\gamma_i}$. Each of these $\tilde{\gamma_i}$ projects to M a geodesic γ_i connecting x and y.

If the point x is sufficiently close to γ, then there must be infinitely many γ_i such that each of them wraps around γ in the same direction as the asymptotic geodesic c. We see that no two of these $\gamma_i's$ can intersect at points other than x and y. This is because if γ_i and γ_j do intersect, then their lifts $\tilde{\gamma_i}$ and $\tilde{\gamma_j}$ will intersect before they hit y_i and y_j respectively. This contradicts the fact that two minimal geodesics originating from the same point will not be minimal beyond their first point of intersection. For the same reason, each of these geodesic γ_i can only intersect c at x.

So let us fix x and y as above and show that $B(x, y) = \infty$. If not, let us assume $\{z_i\}$ to be a finite set of points that block all geodesics $\{\gamma_i\}$. Throw away several points from $\{z_i\}$ if necessary, we could assume that all z_i are not on γ. Hence there exists a neighborhood U of γ in M such that no z_i belongs to U. As mentioned above the geodesic c enters and stay in U after a finite length, and since each γ_i cannot intersect c before it hits the point y, there is a uniform t_0 such that for each i, $\gamma_i(t)$ is in U for all $t > t_0$.

So the infinite set of geodesics $\gamma_i|_{[0,t_0]}$, which only intersect at x, have to be blocked by a finite set of points $\{z_i\}$. This is a contradiction and the lemma is proven.$\Box$

Our next tool is monotone twist maps, which can be used to analyze geodesic behaviors on a 2-dimensional torus, details of monotone twist maps will be covered in the next section. For now let us see how the previous lemma can prove a weaker result concerning the blocking properties of a 2-dimensional torus.

Let us now assume that (M, g) is a 2-dimensional torus. The metric g is said to be *bumpy* if all periodic geodesics of (M, g) are non-degenerate. We will now show that a bumpy torus has infinite blocking number.

Proposition 4.7 *If (M, g) is a bumpy torus, then it has infinite blocking number.*

Proof: Firstly, let γ be a shortest periodic geodesic of M. Then γ is a hyperbolic geodesic due to Morse [1]. This implies that γ is isolated. Now we pick another periodic minimizing geodesic γ' in the same homotopy class, such that there is no periodic geodesic from this homotopy class lies in the annulus bounded by γ and γ'. This is possible since γ is isolated, if γ is the only closed geodesic in the homotopy class that we can consider $\gamma' = \gamma$ such that the lift of γ' is next to the lift of γ. Now we observe that the annulus bounded by γ and γ' does not contain any periodic minimal geodesics since geodesics from all other homotopy class must either intersect γ or itself, but we know that a geodesic that intersect itself cannot be minimizing. Therefore we conclude that for the strip in $\mathbb{R}^2$ between the lift of γ and γ', there cannot be a curve which is the lift of a periodic minimal geodesic.

Using monotone twist maps, Bangert [8] [Th 6.8] proved that there exists a minimal geodesic such that c is ω-asymptotic to γ. So in particular, if U is a neighborhood of γ in T, c would stay inside U after a finite length. By Lemma 4.6 we can conclude that (M, g) cannot have finite blocking number.$\square$

Note that Proposition 4.7 is an immediate corollary of our main result.

5 Proof of The Main Result

We now furnish the proof of Theorem 3.1. In view of Theorem 3.2, we could separate the surfaces in term of their genus $\mathfrak{g}$. We will first prove that when $\mathfrak{g} = 1$, the only case that M has finite blocking number is the flat torus. We will then prove that when $\mathfrak{g} > 1$, M cannot have finite blocking number.

5.1 genus $\mathfrak{g} = 1$

Let (M, g) be a topological 2-dimensional torus, we want to show that if (M, g) has finite blocking number, than g must be a flat metric.

Our approach is the following: We will argue by first assuming that (M, g) has finite blocking number, next for each free homotopy class $[\alpha]$ of M, we call an annulus *bad* if the annulus is bounded by two periodic minimal geodesics from $[\alpha]$ such that no other periodic geodesics exists in the annulus. We will show that no bad annulus exists in (M, g). Afterwards we prove that this means M can be foliated by periodic minimal geodesics of the class $[\alpha]$. This in turns will imply (M, g) has no conjugate points and so by Hopf's Theorem g must be a flat metric.

Let us begin by introducing some notations. If $\gamma(s)$ is a periodic minimal geodesic in M, then its lift $\tilde{\gamma}(s)$ is a minimal geodesic in $(\mathbb{R}^2, \tilde{g})$. Let us write

$\tilde{\gamma}(s) = (\xi(s), \eta(s))$ where ξ and η represents the coordinates of $\tilde{\gamma}$ in $\mathbb{R}^2$. We can define the *average slope* α of $\tilde{\gamma}$ by setting $\alpha(\tilde{\gamma}) := \lim_{|s|\to\infty} \eta(s)/\xi(s)$.

Here is a property of the functional α proven in [8]:

1) If $\xi(s)$ is surjective then $\alpha(\tilde{\gamma})$ exists in $(-\infty, \infty)$. If $\xi(s)$ is not surjective then $\xi(s)$ is bounded and we will define $\alpha(\tilde{\gamma}) = \infty$.

Definition: Let $(q,p) \in \mathbb{Z}^2$, denote by $T_{(q,p)}(x)$ the action of the group $\mathbb{Z}^2$ on $\mathbb{R}^2$ that translate the point x by (q,p).

Definition: A lifted periodic geodesic $\tilde{\gamma}$ has *period* $(q,p) \in \mathbb{Z}^2 - \{0\}$ if the translation $T_{(q,p)}\tilde{\gamma}$ and $\tilde{\gamma}$ coincide up to parametrization. The *minimal period* of a geodesic is the pair (q,p) such that the geodesic has period (q,p) and p,q are relatively prime. Obviously, if a periodic minimal geodesic $\tilde{\gamma}$ has period (q,p), then $\alpha(\tilde{\gamma}) = p/q$.

Note that if two periodic minimal geodesics are in the same homotopy class then their lifts have the same minimal period. It is also easy to see that two distinct periodic minimal geodesics with the same period do not intersect [6.6 of [8]]. In the following, we will assume a periodic geodesics γ are equivalent to its iterates $n\gamma, n \in \mathbb{N}$.

Now we are ready to show that a torus with finite blocking number is flat. Firstly let us state a lemma that is similar to Lemma 4.6.

Lemma 5.1 *If (M,g) has two periodic minimal geodesics from the same homotopy class, such that the annulus bounded between does not contain other periodic minimal geodesic, i.e. if M has a bad annulus, then the blocking number of (M,g) is infinite.*

Proof: Denote one of the periodic minimal geodesics by γ. According to 6.8 of [8], there exists a minimal geodesic such that c is ω-asymptotic to γ. So in particular, c is asymptotic to γ according to Definition 4.4. Therefore we can apply Lemma 4.6 to conclude that the blocking number of (M,g) is infinite.$\square$

Note that in the above lemma, we can replace the phrase 'does not contain other periodic minimal geodesics' by 'does not contain other periodic minimal geodesics of the same homotopy class'. It is because periodic geodesics of other homotopy classes must intersect either one of the boundaries of the annulus or is a higher iteration of the geodesic itself.

Let $[\alpha] \in \pi_1(M)$, by the first variation formula we know that there is at least one periodic minimal geodesic. The following proposition shows that if no periodic minimal geodesic in $[\alpha]$ is isolated, then M is foliated by periodic

minimal geodesics from this homotopy class.

Proposition 5.2 *Fix $[\alpha] \in \pi_1(M)$, if for any two periodic minimal geodesics γ and γ' of $[\alpha]$ there is a periodic minimal geodesic of $[\alpha]$ that lies in the annulus bounded by γ and γ', then M is foliated by periodic geodesics in $[\alpha]$.*

Proof: Let γ be the shortest periodic geodesic from the free homotopy class $[\alpha]$, this closed geodesic lifts to $(\mathbb{R}^2, \tilde{g})$ to a minimal geodesic $\tilde{\gamma}$. Let γ' be another periodic minimal geodesic from $[\alpha]$ and $\tilde{\gamma}'$ be the corresponding lift. If there is no such geodesic then we let $\gamma = \gamma'$ and $\tilde{\gamma}'$ be the minimal geodesic in $\mathbb{R}^2$ neighboring $\tilde{\gamma}$ and projects to γ. Next we assert the following is true.

Claim: For any point x in the strip bounded by $\tilde{\gamma}$ and $\tilde{\gamma}'$, there is a minimal geodesic c_x passing through x such that $\alpha(c_x) = \alpha(\gamma) = \alpha(\gamma')$.

PROOF of Claim: To prove this claim, assume that the minimal geodesics $\tilde{\gamma}$ and $\tilde{\gamma}'$ have minimal period (q,p). Let $x \in \mathbb{R}^2$ be a point that lies in the strip between them. Denote by $x_1 = T_{(q,p)}x$, and $x_{i+1} = T_{(q,p)}x_i$, $\forall i \in \mathbb{N}$. Then we can connect each x_i to x by one minimal geodesic c_i.

Note that each c_i cannot touch or cross each of $\tilde{\gamma}$ and $\tilde{\gamma}'$ transversely . It is because if c_i crosses say, $\tilde{\gamma}$ transversely, then it has to cross it at least twice, and we know that geodesics that intersect each other twice cannot be minimal beyond the intersections. If c_i touches $\tilde{\gamma}$ then this will contradict the uniqueness of geodesic for a given initial point and tangent vector. This means all c_i stay in the strip.

Now, let v_i be a vector in U_xM, the unit tangent sphere at x, such that $v_i = c_i'(0)$. Then $\{v_i\}$ is a set of vectors in the compact sphere. So there is a limit $v = \lim_{i\to\infty} v_i$. Let $c(s)$ be the forward complete geodesic satisfying $c(0) = x$, $c'(0) = v$. We now show $c(s)$ stays the strip for all $s > 0$. Suppose not, say, $c(s)$ intersects $\tilde{\gamma}$ transversely at some point. Then there is an $\epsilon > 0$ and $s_0 > 0$ such that $c(s_0)$ is not in the strip and lay outside of the ϵ-neighborhood of $\tilde{\gamma}$.

Since a geodesic is a solution of a second order ODE, the solutions with initial point x continuously depend on the initial vector v. Also as M is geodesic complete, the geodesic flow $\phi_s(x,v)$ on the unit tangent bundle of T is defined for all $s > 0$. Let $f : UM \to T$ be the composition of the time-s_0 map restricted to U_xM, $\phi_{s_0}(x,\cdot) : U_xM \to UM$ with the projection $\pi : UM \to M$ given by $(x,v) \mapsto x$, i.e. $f = \pi \circ \phi_{s_0}$. Then f continuously depends on v. So for the geodesic c there exists a $\delta > 0$ such that for all geodesics $\bar{c}$ with $\bar{c}(0) = x$ and $\bar{c}'(0) = \bar{v}$ for $\|v - \bar{v}\| \leq \delta$, we have $\tilde{d}(\bar{c}(s_0) - c(s_0)) \leq \epsilon$. Recall that $v = \lim_{i\to\infty} v_i$ and c_i is the geodesic at x with initial vector v_i, so by the arguments above there exists arbitrarily long c_i such that $c_i(s_0)$ lies outside the strip. In particular c_i

crosses the boundary of the strip transversely. However this is not possible since c_i connects two points in the strip, if it cross the boundary than it will intersect the boundary minimal geodesic twice, this contradicts the assumption that c_i is a minimal geodesic.

This means that c must stay inside the strip, it is now easy to see that $\alpha(c) = \alpha(\gamma') = \alpha(\gamma)$ and so the claim true.

Now the claim is true so we can then applies Theorem 6.7 of [8], that c is either periodic or is contained in a strip between two periodic minimal geodesics c^- and c^+. For the latter case the strip between c^- and c^+ contains no other periodic minimal geodesic. This will contradict our hypothesis so c can only be periodic. This means each point $x \in M$ lies on a periodic geodesic in $[\alpha]$ and the proposition is proven.$\square$

Now we can finalize the proof of Theorem 3.1 for the case when the genus $\mathfrak{g} = 1$.

Proposition 5.3 *If (M, g) is a 2-dimensional torus with finite blocking number, then (M, g) is flat.*

Proof: Assume that (M, g) has finite blocking number. According to Lemma 5.1, if $[\alpha]$ is a free homotopy class then between any two periodic minimal geodesics of $[\alpha]$ there exists another periodic geodesic. Since the free homotopy class $[\alpha]$ is arbitrarily, Proposition 5.2 implies that periodic minimal geodesics in any fixed free homotopy class foliate M. So according to Innami [Corollary 3.2] [9] (M, g) does not have conjugate points and so by Hopf's Theorem, g must be a flat metric.$\square$

5.2 genus $\mathfrak{g} > 1$

We now let M be a surface of genus $\mathfrak{g}$, where $\mathfrak{g} \geq 2$. Let g be any metric on T. We will now show that (M, g) must have infinite blocking number:

We will make use of Lemma 4.6 again, which states that if a closed surface has a periodic minimal geodesic and another minimal geodesic asymptotic to it, then the surface has infinite blocking number.

Definition 5.4 *Let γ be a closed geodesic on M. A* collar *of γ is the image of a diffeomorphism $f : \mathbb{S}^1 \times [0, a] \to M$ with $f(\mathbb{S}^1 \times \{0\}) = \gamma$, $a \in \mathbb{R}^+$.*

In the following, a *periodic minimal geodesic* is a closed geodesics with minimal length in the homotopy class. A *cylinder* is a manifold diffeomorphic to the

product of an interval (open or closed) and $\mathbb{S}^1$.

We will show that a surface of higher genus has a closed geodesic and a collar, such that the collar contains an asymptotic geodesic. Once this is established the proof of our desired result is trivial.

Let $[\alpha]$ be a homotopy class of M which represents the cross section of a 'handle' of M. Using the variation formula we know there is at least one periodic minimal geodesic $\gamma \in [\alpha]$. As a matter of fact, periodic minimal geodesics of $[\alpha]$ are well ordered:

Lemma 5.5 *Periodic minimal geodesics of $[\alpha]$ form a totally ordered set.*

Proof: Recall that no two periodic minimal geodesics from $[\alpha]$ intersect, this is because if they do, then their lifts to the universal cover will intersect each other more than one time, this contradicts the assumption that both geodesics are minimal. Since we also assume that the genus of M is greater than 1, closed geodesics from $[\alpha]$ cannot homotopically pass the portion of M containing other handles. Therefore we conclude that periodic minimal geodesics of $[\alpha]$ form a totally ordered set.$\square$

Proposition 5.6 *If M is a surface of genus $g \geq 2$, then there exists a closed geodesic γ and a collar such that except γ, there is no closed geodesics homotopic to γ that intersect the collar.*

Proof: As above, let $[\alpha]$ be a homotopy class of M which represents the cross section of a 'handle' of M. Our claim is that there is a periodic minimal geodesic of $[\alpha]$ with a collar such that no geodesics of $[\alpha]$ that intersect the collar.

By Lemma 5.4, periodic minimal geodesics of $[\alpha]$ form a totally ordered set. We also know that periodic minimal geodesics are the critical points of the length functional, therefore the set of minimal geodesics is closed. Hence with respect to the ordering on minimal geodesics of the class $[\alpha]$, there is a 'maximum' γ and 'minimum' γ' so that all periodic minimal geodesics of $[\alpha]$ lies in the cylinder bounded by γ and γ'.

If γ and γ' are the same geodesic, there could be two possibilities. The first one is that there is only one periodic minimal geodesic γ in the class $[\alpha]$, in this case we can find a neighborhood of γ such that its retract is γ. Since there is only one minimal geodesic, the neighborhood contains a collar, and the only periodic minimal geodesic contained in the collar is γ itself, thus the statement of the Proposition is true in this case.

The second possibility is that there are more than one periodic minimal geodesic. In this case if γ and γ' are the same geodesic, then the geodesic γ has to homotope along the handle and overlap with itself. However since the genus of M is strictly greater than 1, this cannot be done. Therefore we conclude that for the case where γ and γ' are the same geodesic, there could only be one periodic minimal geodesic and we are done.

Now suppose that γ and γ' are two distinct periodic minimal geodesics. Note again that γ cannot intersect or touch γ'. Hence if we denote by K the cylinder bounded by γ and γ', there is a neighborhood U of K that is diffeomorphic to a cylinder. Now since there is no periodic minimal geodesics from $[\alpha]$ outside of K, the set $(U \cup \gamma) \backslash K$ contains a closed geodesic γ and a collar such that except γ, there is no closed geodesics homotopic to γ that intersect the collar, finishing the proof of the Proposition. $\square$

Next proposition will direct us to the proof of higher genus case of Theorem 3.1:

Proposition 5.7 *If M is a surface of genus $g > 1$, then M has a closed minimal geodesic and another minimal geodesic asymptotic to it.*

Proof: According to Proposition 5.5, M has a periodic minimal geodesic γ and a collar C such that except γ, there is no closed geodesics homotopic to γ that intersects C.

Note again that C is diffeomorphic to a cylinder. We now cut an open neighborhood U of C out such that U is diffeomorphic to a cylinder, then γ is the shortest closed geodesic in U. After that we smoothly glue a closed Riemannian cylinder to U, we would then obtain a 2 dimensional Riemannian torus.

Since γ is the shortest closed curve in the U, we can glue the cylinder to U such that γ remains to be the shortest closed geodesic in the homotopy class. For instance, let the metric of the cylinder be that all closed curves in it have lengths not shorter than γ, which is possible since the length of each boundary of U is not shorter than γ. This way we can guarantee that γ, as a closed curve of the torus, is the shortest in its free homotopy class.

We know that the geodesic behaviors of a surface is completely determined by the metric. In particular, the geodesic behaviors of C as a subset of M is the same as that of C as a subset of the glued 2 dimensional torus. Now as a subset of the glued torus, C does not contain any periodic minimal geodesics homotopic to γ. If we consider the universal cover $\mathbb{R}^2$, γ is lifted to a infinitely long minimal geodesic $\tilde{\gamma}$ and the collar C is lifted to a strip $\tilde{C}$. Let $\tilde{\gamma}'$ be the lift of an adjacent periodic minimal geodesic from the same homotopy class such that $\tilde{\gamma}$ and $\tilde{\gamma}'$ bound $\tilde{C}$. This could be either one of the followings:

a) $\tilde{\gamma}'$ is the lift of a distinct periodic minimal geodesic.

b)$\tilde{\gamma}'$ is another lift of γ which is adjacent to $\tilde{\gamma}$.

In case **a** we claim that $\tilde{\gamma}'$ cannot intersect $\tilde{C}$. To see why this is true, let us first remark that periodic minimal geodesics of a torus are exactly the lifts of closed geodesics which have minimal length in their free homotopy class, see 6.6 [8].

If γ_1 is a closed curve homotopic to γ that lies completely in C, then the length of γ_1 is strictly greater than the length of γ, this is because there is no other periodic minimal geodesics in C.

Now we assume that γ_1 is a closed curve partially lies in C. Note that in the gluing process we ensured that all closed curve outside C is longer than γ. Together with the facts that both boundaries of C is not shorter than γ, we see that γ_1 cannot be shorter than γ.

What matters in both case **a** and **b** is that $\tilde{\gamma}'$ and $\tilde{\gamma}$ bounds a strip that contains $\tilde{C}$, and the strip does not contain any periodic minimal geodesics of the same homotopy class. So we can apply Theorem 6.8 of [8] to show that there is a minimal geodesic c in the strip which is ω-asymptotic to $\tilde{\gamma}$. Now we consider the portion of the minimal geodesic c which stays in $\tilde{C}$, this geodesic projects to a minimal geodesic asymptotic to γ. Finally, since the geodesic behaviors of an area is determined locally by the metric, therefore if we revert the cutting and gluing procedures that changes M to a 2 dimensional torus, the geodesic c staying in the collar C remains exactly the same. Therefore on the surface M, c is a minimal geodesic asymptotic to the periodic minimal geodesic γ and the Proposition is proven.□

With the above propositions and lemma, we can now prove the following proposition:

Proposition 5.8 *If M is a closed Riemannian surface with genus greater than or equal to 2, then the blocking number of M is infinite.*

Proof: By Proposition 5.6, M has a closed minimal geodesic and another minimal geodesic asymptotic to it. Therefore we can apply Lemma 4.6 to conclude that M has infinite blocking number. □

Combining the statements of Propositions 5.3 and 5.7, and using Theorem 3.2, we have proven Theorem 3.1.

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